

Independent sets in random graphs from the weighted second moment method

Varsha Dani
 Computer Science Department
 University of New Mexico

Cristopher Moore
 Computer Science Department
 University of New Mexico
 and the Santa Fe Institute

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Abstract

We prove new lower bounds on the likely size of an independent set in a random graph with a given average degree. Our method is a weighted version of the second moment method, where we give each independent set a weight based on the total degree of its vertices.

1 Introduction

We are interested in the likely size of the largest independent set S in a random graph with a given average degree. Azuma's inequality implies that $|S|$ is tightly concentrated around its expectation. Moreover, it is easy to see that $|S| = \Theta(n)$ whenever the average degree is constant. Thus for each constant c , there is a constant $\alpha_{\text{crit}} = \alpha_{\text{crit}}(c)$ such that

$$\Pr[G(n, p = c/n) \text{ has an independent set of size } \alpha n] = \begin{cases} 1 & \alpha < \alpha_{\text{crit}} \\ 0 & \alpha > \alpha_{\text{crit}}. \end{cases}$$

By standard arguments this holds in $G(n, m = cn/2)$ as well.

Our goal is to bound α_{crit} as a function of c , or equivalently to bound

$$c_{\text{crit}} = \sup \{c : \alpha_{\text{crit}}(c) \geq \alpha\},$$

as a function of α . For $c \leq e$, a greedy algorithm of Karp and Sipser [3] asymptotically finds a maximal independent set, and analyzing this algorithm with differential equations yields the exact value of α_{crit} . For larger c , Frieze [1] determined α_{crit} to within $o(1/c)$, where o refers to the limit where c is large. These bounds were improved by Coja-Oghlan and Efthymiou [2] who prove detailed results on the structure of the set of independent sets.

We improve these bounds significantly. Our method is a weighted version of the second moment method, inspired by the work of Achlioptas and Peres [4] on random k -SAT, where each independent set is given a weight depending on the total degree of its vertices.

We work in a modified version of the $G(n, m)$ model which we call $\tilde{G}(n, m)$. For each of the m edges, we choose two vertices u, v uniformly and independently and connect them. This may lead to a few multiple edges or self-loops, and any vertex with a self-loop cannot belong to an independent set. In the sparse case where $m = cn/2$ for constant c , with constant positive probability $\tilde{G}(n, m = cn/2)$ has no multiple edges or self-loops, in which case it is uniform in the usual model $G(n, m = cn/2)$ where edges are chosen without replacement from distinct pairs of vertices. Thus any property which holds with high probability for $\tilde{G}(n, m)$ also holds with high probability for $G(n, m)$, and any bounds we prove on α_{crit} in $\tilde{G}(n, m)$ also hold in $G(n, m)$.

We review the first moment upper bound on α_{crit} from Bollobás [5]. Let X denote the number of independent sets of size αn in $\tilde{G}(n, m)$. Then

$$\Pr[X > 0] \leq \mathbb{E}[X].$$

By linearity of expectation, $\mathbb{E}[X]$ is the sum over all $\binom{n}{\alpha n}$ sets of αn vertices of the probability that a given one is independent. The m edges (u, v) are chosen independently and for each one $u, v \in S$ with probability α^2 , so

$$\mathbb{E}[X] = \binom{n}{\alpha n} (1 - \alpha^2)^m.$$

In the limit $n \rightarrow \infty$, Stirling's approximation $n! = (1 + o(1))\sqrt{2\pi n} n^n e^{-n}$ gives

$$\binom{n}{\alpha n} \sim \frac{1}{\sqrt{n}} e^{nh(\alpha)},$$

where h is the entropy function

$$h(\alpha) = -\alpha \ln \alpha - (1 - \alpha) \ln(1 - \alpha),$$

and where \sim hides constants that depend smoothly on α . Thus

$$\mathbb{E}[X] \sim e^{n(h(\alpha) + (c/2) \ln(1 - \alpha^2))}.$$

For each c , the α such that

$$h(\alpha) + (c/2) \ln(1 - \alpha^2) = 0 \tag{1}$$

is an upper bound on $\alpha_{\text{crit}}(c)$, since for larger α the expectation $\mathbb{E}[X]$ is exponentially small.

We find it more convenient to parametrize our bounds in terms of the $c_{\text{crit}}(\alpha)$. Then (1) gives the following upper bound,

$$c_{\text{crit}}(\alpha) \leq 2 \frac{\alpha \ln \alpha + (1 - \alpha) \ln(1 - \alpha)}{\ln(1 - \alpha^2)} \leq 2 \frac{\ln(1/\alpha) + 1}{\alpha}. \tag{2}$$

We will prove the following nearly-matching lower bound.

Theorem 1. *For any constant $x > 4/e$, for sufficiently small α*

$$c_{\text{crit}}(\alpha) \geq 2 \frac{\ln(1/\alpha) + 1}{\alpha} - \frac{x}{\sqrt{\alpha}}. \tag{3}$$

We note that Coja-Oghlan and Efthymiou [2] bounded c_{crit} within a slightly larger factor $O(\sqrt{\ln(1/\alpha)/\alpha})$.

Inverting (2) and Theorem 1 gives the following bounds on $\alpha_{\text{crit}}(c)$. The lower bound is a significant improvement over previous results:

Corollary 2. *Let $W(z)$ denote the largest positive root y of the equation $ye^y = z$. Then for any constant $y > 4\sqrt{2}/e$,*

$$\frac{2}{c} W\left(\frac{ec}{2}\right) - y \frac{\sqrt{\ln c}}{c^{3/2}} \leq \alpha_{\text{crit}} \leq \frac{2}{c} W\left(\frac{ec}{2}\right),$$

where the lower bound holds for sufficiently large c .

If we like we can expand $W(ec/2)$ asymptotically in c ,

$$\begin{aligned} W\left(\frac{ec}{2}\right) &= \ln c - \ln \ln c + 1 - \ln 2 + \frac{\ln \ln c}{\ln c} - \frac{1 - \ln 2}{\ln c} \\ &\quad + \frac{1}{2} \frac{(\ln \ln c)^2}{(\ln c)^2} - (2 - \ln 2) \frac{\ln \ln c}{(\ln c)^2} + \frac{3 + (\ln 2)^2 - 4 \ln 2}{2(\ln c)^2} + O\left(\frac{(\ln \ln c)^3}{(\ln c)^3}\right). \end{aligned}$$

The first few of these terms correspond to the bound in [1], and we can extract as many additional terms as we wish.

2 The weighted second moment method

Our proof uses the second moment method. For any nonnegative random variable X , the Cauchy-Schwarz inequality implies that

$$\Pr[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}. \quad (4)$$

Unfortunately, applying this directly to the number X of independent sets fails utterly. The problem is that for most pairs of sets of size αn , the events that they are independent are highly correlated, unlike the case where the average degree grows sufficiently quickly with n [1, 5]. As a result, $\mathbb{E}[X^2]$ is exponentially larger than $\mathbb{E}[X]^2$, and the second moment method yields an exponentially small lower bound on $\Pr[X > 0]$.

One way to do away with these correlations, used by Frieze in [1], is to partition the vertices into sets V_i of $\lfloor 1/\alpha \rfloor$ vertices each and focus on those independent sets that intersect each V_i exactly once. Here we pursue a different approach, inspired by the work of Achlioptas and Peres [4] on random k -SAT. The idea is to give each independent set S a weight $w(S)$, depending exponentially on local quantities in the graph. Specifically, we define

$$w(S) = \mu^{\# \text{ of edges } (u, v) \text{ with } u, v \notin S},$$

where $\mu < 1$. If the number of edges m is fixed, the number of edges where neither endpoint is in S is simply m minus the total degree of the vertices in S . Thus we can also write

$$w(S) = \mu^{m - \sum_{v \in S} \deg(v)}.$$

Weighting each S in this way counteracts the temptation for S to consist primarily of vertices of low degree. This is analogous to [4], where satisfying assignments are given a weight that discourages them from satisfying the majority of literals in the formula.

We will apply the second moment method to the random variable

$$X = \sum_{\substack{S \subseteq V, |S| = \alpha n \\ S \text{ independent}}} w(S).$$

If we tune μ properly, then for particular α^*, c^* we have $\mathbb{E}[X^2] \sim \mathbb{E}[X]^2$, in which case $\Pr[X > 0]$ is bounded above zero. In that case $c_{\text{crit}}(\alpha^*) \geq c^*$, or equivalently $\alpha_{\text{crit}}(c^*) \geq \alpha^*$.

To this end, let us compute the first and second moments of our random variable X . We extend the weight function $w(S)$ to all sets $S \subseteq V$ by setting $w(S) = 0$ if S is not independent. That is,

$$X = \sum_{\substack{S \subseteq V \\ |S| = \alpha n}} w(S)$$

where

$$w(S) = \prod_{(u, v) \in E} w_{u, v}(S)$$

and

$$w_{u, v}(S) = \begin{cases} \mu & \text{if } u, v \notin S \\ 1 & \text{if } u \in S, v \notin S \text{ or vice versa} \\ 0 & \text{if } u, v \in S. \end{cases} \quad (5)$$

We start by computing $\mathbb{E}[X]$. Fix a set S of size αn . Since the m edges are chosen independently,

$$\mathbb{E}[w(S)] = w_1(\alpha, \mu)^m \quad \text{where } w_1(\alpha, \mu) = \mathbb{E}_{u, v}[w_{u, v}(S)].$$

For each edge (u, v) in $\tilde{G}(n, m)$, u and v are chosen randomly and independently, so the probabilities of the three cases in (5) are $(1 - \alpha)^2$, $2\alpha(1 - \alpha)$, and α^2 respectively. Thus

$$w_1(\alpha, \mu) = (1 - \alpha)^2 \mu + 2\alpha(1 - \alpha).$$

By linearity of expectation,

$$\mathbb{E}[X] = \sum_{\substack{S \subseteq V \\ |S|=\alpha n}} \mathbb{E}[w(S)] = \binom{n}{\alpha n} w_1(\alpha, \mu)^m.$$

Using Stirling's approximation, $\binom{n}{\alpha n} \sim \frac{1}{\sqrt{n}} e^{nh(\alpha)}$ and substituting $m = cn/2$ gives

$$\mathbb{E}[X] \sim \frac{1}{\sqrt{n}} e^{nf_1(\alpha)} \quad \text{where} \quad f_1(\alpha) = h(\alpha) + \frac{c}{2} \ln w_1(\alpha, \mu). \quad (6)$$

As before, \sim hides constant factors that depend smoothly on α .

Next we compute the second moment. We have

$$\mathbb{E}[X^2] = \mathbb{E} \left[\sum_S w(S) \sum_T w(T) \right] = \sum_{S,T} \mathbb{E}[w(S) w(T)]$$

where S and T are subsets of V of size αn . The expectation of $w(S) w(T)$ does not depend on the specific choice of S and T , but it does depend on the size of their intersection. We say that S and T have *overlap* ζ if $|S \cap T| = \zeta n$. Again using the independence of the edges, we have

$$\mathbb{E}[w(S) w(T)] = w_2(\alpha, \zeta, \mu)^m \quad \text{where} \quad w_2(\alpha, \zeta, \mu) = \mathbb{E}_{u,v} [w_{u,v}(S) w_{u,v}(T)].$$

For each edge (u, v) of \tilde{G} , the probability that it has no endpoints in S or T is $(1 - 2\alpha + \zeta)^2$, in which case it contributes μ^2 to $w_{u,v}(S) w_{u,v}(T)$. The probability that it has one endpoint in S and none in T or vice versa is $2(2\alpha - 2\zeta)(1 - 2\alpha + \zeta)$, in which case it contributes μ . Finally, the probability that it has one endpoint in S and one in T is $2(\alpha - \zeta)^2 + 2\zeta(1 - 2\alpha + \zeta)$, in which case it contributes 1. With the remaining probability it has both endpoints in S or T , in causing them to be non-independent and contributing zero. Thus

$$w_2(\alpha, \zeta, \mu) = (1 - 2\alpha + \zeta)^2 \mu^2 + 4(\alpha - \zeta)(1 - 2\alpha + \zeta)\mu + 2(\alpha - \zeta)^2 + 2\zeta(1 - 2\alpha + \zeta)$$

Observe that when $\zeta = \alpha^2$, as it typically would be if S and T were chosen independently and uniformly, we have

$$w_2 = w_1^2. \quad (7)$$

The number of pairs of sets S, T of size αn and intersection of size $z = \zeta n$ is the multinomial

$$\binom{n}{\zeta n, (\alpha - \zeta)n, (\alpha - \zeta)n, (1 - 2\alpha + \zeta)n} = \binom{n}{\alpha n} \binom{\alpha n}{\zeta n} \binom{(1 - \alpha)n}{(\alpha - \zeta)n},$$

and linearity of expectation gives

$$\mathbb{E}[X^2] = \sum_{z=0}^{\alpha n} \binom{n}{z, \alpha n - z, \alpha n - z, (1 - 2\alpha)n + z} w_2(\alpha, \zeta, \mu)^m.$$

This sum is dominated by the terms where $\zeta = z/n$ is bounded inside the interval $(0, \alpha)$. Stirling's approximation then gives

$$\binom{n}{\zeta n, (\alpha - \zeta)n, (\alpha - \zeta)n, (1 - 2\alpha + \zeta)n} \sim \frac{1}{n^{3/2}} e^{n[h(\alpha) + \alpha h(\zeta/\alpha) + (1 - \alpha)h(\frac{\alpha - \zeta}{1 - \alpha})]}, \quad (8)$$

where \sim hides constants that vary slowly with α and ζ . Thus the contribution to $\mathbb{E}[X^2]$ of pairs of sets with overlap $\zeta \in (0, \alpha)$ is

$$\frac{1}{n^{3/2}} e^{nf_2(\alpha, \zeta, \mu)} \quad \text{where} \quad f_2(\alpha, \zeta, \mu) = h(\alpha) + \alpha h\left(\frac{\zeta}{\alpha}\right) + (1 - \alpha)h\left(\frac{\alpha - \zeta}{1 - \alpha}\right) + \frac{c}{2} \ln w_2(\alpha, \zeta, \mu). \quad (9)$$

Combining this with (6), we can write

$$\frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} \sim \frac{1}{\sqrt{n}} \sum_{z=0}^{\alpha n} e^{n\phi(z/n)}, \quad (10)$$

where

$$\phi(\zeta) = f_2(\alpha, \zeta, \mu) - 2f_1(\alpha, \mu) = \alpha h\left(\frac{\zeta}{\alpha}\right) + (1-\alpha) h\left(\frac{\alpha-\zeta}{1-\alpha}\right) - h(\alpha) + \frac{c}{2} \ln \frac{w_2(\alpha, \zeta, \mu)}{w_1(\alpha, \mu)^2}.$$

Using (7) and the fact that the entropy terms cancel, we have

$$\phi(\alpha^2) = 0.$$

In other words, the contribution to $\mathbb{E}[X^2]$ from pairs of sets with overlap α^2 is proportional to $\mathbb{E}[X]^2$.

We can now replace the sum in (10) with an integral,

$$\frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} \sim \frac{1}{\sqrt{n}} \sum_{z=0}^{\alpha n} e^{n\phi(z/n)} \sim \sqrt{n} \int_0^\alpha e^{n\phi(\zeta)} d\zeta,$$

and evaluate this integral using Laplace's method as in [6, Lemma 3]. Its asymptotic behavior depends on the maximum value of ϕ ,

$$\phi_{\max} = \max_{\zeta \in [0, \alpha]} \phi(\zeta).$$

If $\phi'' < 0$ at the corresponding ζ_{\max} , then it is dominated by an interval of width $\Theta(1/\sqrt{n})$ around ζ_{\max} and

$$\frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} \sim e^{n\phi_{\max}}.$$

If $\phi_{\max} = \phi(\alpha^2) = 0$, then $\mathbb{E}[X^2] \sim \mathbb{E}[X]^2$ and the second moment method succeeds. Thus our goal is to show that ϕ is maximized at α^2 .

For this to happen, we at least need $\zeta = \alpha^2$ to be a *local* maximum of ϕ . In particular, we need

$$\phi'(\alpha^2) = 0. \quad (11)$$

Differentiating, we find that (11) holds if

$$\mu = \frac{1-2\alpha}{1-\alpha}.$$

Henceforth, we will fix μ to this value. In that case we have

$$w_1 = 1 - \alpha \quad \text{and} \quad w_2 = (1-\alpha)^2 + \frac{(\zeta - \alpha^2)^2}{(1-\alpha)^2},$$

so

$$\phi(\zeta) = \alpha h\left(\frac{\zeta}{\alpha}\right) + (1-\alpha) h\left(\frac{\alpha-\zeta}{1-\alpha}\right) - h(\alpha) + \frac{c}{2} \ln \left(1 + \frac{(\zeta - \alpha^2)^2}{(1-\alpha)^4}\right)$$

The remainder of this paper is dedicated to showing that for sufficiently small α as a function of c or vice versa, ϕ is indeed maximized at α^2 .

3 Finding and bounding the maxima

Using $\ln(1+x) \leq x$, we write $\phi(\zeta) \leq \psi(\zeta)$ where

$$\psi(\zeta) = \alpha h\left(\frac{\zeta}{\alpha}\right) + (1-\alpha) h\left(\frac{\alpha-\zeta}{1-\alpha}\right) - h(\alpha) + \frac{c}{2} \frac{(\zeta-\alpha^2)^2}{(1-\alpha)^4}. \quad (12)$$

Note that

$$\psi(\alpha^2) = \phi(\alpha^2) = 0.$$

Our goal is to show for an appropriate c that $\zeta = \alpha^2$ is in fact the global maximum of ψ , and therefore of ϕ . In what follows, asymptotic symbols such as O and o refer to the limit $\alpha \rightarrow 0$, or equivalently the limit $c \rightarrow \infty$. Error terms may be positive or negative unless otherwise stated.

The first two derivatives of $\psi(\zeta)$ are

$$\psi'(\zeta) = \frac{c(\zeta-\alpha^2)}{(1-\alpha)^4} + 2\ln(\alpha-\zeta) - \ln\zeta - \ln(1-2\alpha+\zeta) \quad (13)$$

$$\psi''(\zeta) = \frac{c}{(1-\alpha)^4} - \frac{2}{\alpha-\zeta} - \frac{1}{\zeta} - \frac{1}{1-2\alpha+\zeta} \quad (14)$$

The second derivative $\psi''(\zeta)$ tends to $-\infty$ at $\zeta = 0$ and $\zeta = \alpha$. Setting $\psi''(\zeta) = 0$ yields a cubic equation in ζ which has one negative root and, for sufficiently small α , two positive roots in the interval $[0, \alpha]$. Thus for each α and sufficiently small α , there are $0 < \zeta_1 < \zeta_2 < \alpha$ where

$$\psi''(\zeta) \begin{cases} < 0 & 0 < \zeta < \zeta_1 \\ > 0 & \zeta_1 < \zeta < \zeta_2 \\ < 0 & \zeta_2 < \zeta < \alpha. \end{cases}$$

It follows that ψ can have at most two local maxima. One is in the interval $[0, \zeta_1]$, and the following lemma shows that for the relevant α and c this is α^2 :

Lemma 3. *If $c = o(1/\alpha^2)$ then for sufficiently small α , $\zeta_1 > \alpha^2$ and $\psi(\alpha^2)$ is a local maximum.*

The other local maximum is in the interval $[\zeta_2, \alpha]$, and we denote it ζ_3 . To locate it, first we bound ζ_2 :

Lemma 4. *If*

$$c = (2 + o(1)) \frac{\ln(1/\alpha)}{\alpha},$$

then

$$\frac{\zeta_2}{\alpha} = 1 - \delta_2 \quad \text{where} \quad \delta_2 = \frac{1 + o(1)}{\ln(1/\alpha)}.$$

Thus ζ_2/α , and therefore ζ_3/α , tends toward 1 as $\alpha \rightarrow 0$.

We can now locate ζ_3 when α is close to its critical value.

Lemma 5. *If*

$$c = \frac{1}{\alpha} (2 \ln(1/\alpha) + 2 - o(1)),$$

then

$$\frac{\zeta_3}{\alpha} = 1 - \delta_3 \quad \text{where} \quad \delta_3 = \frac{1 + o(1)}{e} \sqrt{\alpha}.$$

Lemma 6. *For any constant $x > 4/e$, if*

$$c = \frac{2 \ln(1/\alpha) + 2 - x\sqrt{\alpha}}{\alpha},$$

then $\psi(\zeta_3) < 0$ for sufficiently small α .

4 Proofs

Proof of Lemma 3. Setting $\zeta = \alpha^2$ in (14) gives

$$\psi''(\alpha^2) < \frac{c}{(1-\alpha)^4} - \frac{1}{\alpha^2}.$$

If $c = o(1/\alpha^2)$ this is negative for sufficiently small α , in which case $\zeta_1 > \alpha^2$ and $\psi(\alpha^2)$ is a local maximum. \square

Proof of Lemma 4. For any constant b , if

$$\frac{\zeta}{\alpha} = 1 - \delta \quad \text{where} \quad \delta = \frac{b}{\ln(1/\alpha)} \quad (15)$$

then (14) gives

$$\psi''(\zeta) = \left(2 - \frac{2}{b} + o(1)\right) \frac{\ln(1/\alpha)}{\alpha} - O(1/\alpha).$$

If $b \neq 1$, for sufficiently small α this is negative if $b < 1$ and positive if $b > 1$. Therefore $\zeta_2/\alpha = 1 - \delta_3$ where $\delta_3 = (1 + o(1))/\ln(1/\alpha)$. \square

Proof of Lemma 5. Lemma 4 tells us that $\zeta_3 = \alpha(1 - \delta)$ for some

$$\delta < \frac{1 + o(1)}{\ln(1/\alpha)}.$$

Setting $\zeta = \alpha(1 - \delta)$ in (13) and using

$$\frac{1}{(1-\alpha)^4} = 1 + O(\alpha) \quad \text{and} \quad -\ln(1-x) = O(x)$$

gives, after some algebra,

$$\psi'(\zeta) = \alpha c + \ln \alpha + 2 \ln \delta + O(\alpha \delta c) + O(\alpha^2 c).$$

For any constant b , setting

$$\delta = \frac{b\sqrt{\alpha}}{e}$$

gives

$$\psi'(\zeta) = \alpha c + 2 \ln \alpha + 2 \ln b - 2 + O(\alpha^{3/2} c),$$

and setting

$$c = \frac{2 \ln(1/\alpha) + 2 - \varepsilon}{\alpha}$$

then gives

$$\psi'(\zeta) = 2 \ln b - \varepsilon + o(1).$$

If $\varepsilon = o(1)$ and $b \neq 1$, for sufficiently small α this is negative if $b < 1$ and positive if $b > 1$. Therefore $\zeta_3/\alpha = 1 - \delta_3$ where $\delta_3 = (1 + o(1))\sqrt{a}/e$. \square

Proof of Lemma 6. Setting $\zeta = \alpha(1 - \delta)$ where $\delta = b\sqrt{a}/e$ in (12) and using the Taylor series

$$\frac{1}{(1-\alpha)^4} = 1 + 4\alpha + O(\alpha^2) \quad \text{and} \quad -\ln(1-x) = x + x^2/2 + O(x^3)$$

gives, after a fair amount of algebra,

$$\begin{aligned}\psi(\zeta) &= \alpha(\ln \alpha - 1) - \left(\frac{2b \ln \alpha - 4b + 2b \ln b}{e} \right) \alpha^{3/2} + \left(\frac{c+1}{2} - \frac{b^2}{2e^2} \right) \alpha^2 \\ &\quad - \frac{b}{e} \alpha^{5/2} c + \left(\frac{b^2}{2e^2} + 1 \right) \alpha^3 c + O(\alpha^{7/2} c) + O(\alpha^{5/2}).\end{aligned}$$

Setting

$$c = \frac{2 \ln(1/\alpha) + 2 - x\sqrt{\alpha}}{\alpha}$$

for constant x causes the terms proportional to $\alpha \ln \alpha$, α , and $\alpha^{3/2} \ln \alpha$ to cancel, leaving

$$\psi(\zeta) = \left(\frac{2b(1 - \ln b)}{e} - \frac{x}{2} \right) \alpha^{3/2} + O(\alpha^2).$$

The coefficient of $\alpha^{3/2}$ is maximized when $b = 1$, and is negative whenever $x > 4/e$. In that case, $\psi(\zeta_3) < 0$ for sufficiently small α , completing the proof. \square

Proof of Corollary 2. First note that

$$\alpha_0 = \frac{2}{c} W\left(\frac{ec}{2}\right) \tag{16}$$

is the root of the equation

$$c = 2 \frac{\ln(1/\alpha_0) + 1}{\alpha_0},$$

since we can also write it as

$$e^{c\alpha/2} = \frac{e}{\alpha_0},$$

and multiplying both sides by $c\alpha_0/2$ gives

$$\frac{c\alpha_0}{2} e^{c\alpha_0/2} = \frac{ec}{2},$$

in which case (16) follows from the definition of W .

The root α of

$$c = 2 \frac{\ln(1/\alpha) + 1}{\alpha} - \frac{x}{\sqrt{\alpha}}$$

is then at least

$$\frac{2}{c} W\left(\frac{ec}{2}\right) + (x + o(1)) \frac{\partial \alpha_0}{\partial c} \sqrt{\frac{c}{2 \ln c}}$$

since $\alpha = (1 + o(1))2 \ln c/c$ and $\partial^2 \alpha_0 / \partial^2 c \geq 0$. Since

$$\frac{\partial \alpha_0}{\partial c} = -(1 + o(1)) \frac{2 \ln c}{c^2},$$

the statement follows from Theorem 1. \square

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Independent Sets in Random Graphs from the Weighted Second Moment Method

Varsha Dani¹ and Cristopher Moore^{1,2}

¹ Computer Science Department, University of New Mexico

² Santa Fe Institute

Abstract. We prove new lower bounds on the likely size of the maximum independent set in a random graph with a given constant average degree. Our method is a weighted version of the second moment method, where we give each independent set a weight based on the total degree of its vertices.

1 Introduction

We are interested in the likely size of the largest independent set S in a random graph with a given average degree. It is easy to see that $|S| = \Theta(n)$ whenever the average degree is constant, and Shamir and Spencer [8] showed using Azuma's inequality that, for any fixed n , $|S|$ is tightly concentrated around its mean. Moreover, Bayati, Gamarnik, and Tetali [3] recently showed that $|S|/n$ converges to a limit with high probability. Thus for each constant c there is a constant $\alpha_{\text{crit}} = \alpha_{\text{crit}}(c)$ such that

$$\lim_{n \rightarrow \infty} \Pr[G(n, p = c/n) \text{ has an independent set of size } \alpha n] = \begin{cases} 1 & \alpha < \alpha_{\text{crit}} \\ 0 & \alpha > \alpha_{\text{crit}} \end{cases}.$$

By standard arguments this holds in $G(n, m = cn/2)$ as well.

Our goal is to bound α_{crit} as a function of c , or equivalently to bound

$$c_{\text{crit}} = \sup \{c : \alpha_{\text{crit}}(c) \geq \alpha\},$$

as a function of α . For $c \leq e$, a greedy algorithm of Karp and Sipser [7] asymptotically finds a maximal independent set, and analyzing this algorithm with differential equations yields the exact value of α_{crit} . For larger c , Frieze [6] determined α_{crit} to within $o(1/c)$, where o refers to the limit where c is large. These bounds were improved by Coja-Oghlan and Efthymiou [5] who prove detailed results on the structure of the set of independent sets.

We further improve these bounds. Our method is a weighted version of the second moment method, inspired by the work of Achlioptas and Peres [2] on random k -SAT, where each independent set is given a weight depending on the total degree of its vertices. In addition to improving bounds on this particular problem, our hope is that this advances the art and science of inventing random variables that counteract local sources of correlation in random structures.

We work in a modified version of the $G(n, m)$ model which we call $\tilde{G}(n, m)$. For each of the m edges, we choose two vertices u, v uniformly and independently and connect them. This may lead to a few multiple edges or self-loops. A vertex with a self-loop cannot belong to an independent set. In the sparse case where $m = cn/2$ for constant c , with constant positive probability $\tilde{G}(n, m = cn/2)$ has no multiple edges or self-loops, in which case it is uniform in the usual model $G(n, m = cn/2)$ where edges are chosen without replacement from distinct pairs of vertices. Thus any property which holds with high probability for $\tilde{G}(n, m)$ also holds with high probability for $G(n, m)$, and any bounds we prove on α_{crit} in $\tilde{G}(n, m)$ also hold in $G(n, m)$.

We review the first moment upper bound on α_{crit} from Bollobás [4]. Let X denote the number of independent sets of size αn in $\tilde{G}(n, m)$. Then

$$\Pr[X > 0] \leq \mathbb{E}[X].$$

By linearity of expectation, $\mathbb{E}[X]$ is the sum over all $\binom{n}{\alpha n}$ sets of αn vertices of the probability that a given one is independent. The m edges (u, v) are chosen independently and for each one $u, v \in S$ with probability α^2 , so

$$\mathbb{E}[X] = \binom{n}{\alpha n} (1 - \alpha^2)^m.$$

In the limit $n \rightarrow \infty$, Stirling's approximation $n! = (1 + o(1))\sqrt{2\pi n} n^n e^{-n}$ gives

$$\binom{n}{\alpha n} = \Theta\left(\frac{1}{\sqrt{n}} e^{nh(\alpha)}\right), \quad (1)$$

where h is the entropy function

$$h(\alpha) = -\alpha \ln \alpha - (1 - \alpha) \ln(1 - \alpha),$$

and where Θ hides constants that depend smoothly on α . Thus

$$\mathbb{E}[X] = \Theta\left(\frac{1}{\sqrt{n}} e^{n(h(\alpha) + (c/2) \ln(1 - \alpha^2))}\right).$$

For each c , the α such that

$$h(\alpha) + (c/2) \ln(1 - \alpha^2) = 0 \quad (2)$$

is an upper bound on $\alpha_{\text{crit}}(c)$, since for larger α the expectation $\mathbb{E}[X]$ is exponentially small.

We find it more convenient to parametrize our bounds in terms of the function $c_{\text{crit}}(\alpha)$. Then (2) gives the following upper bound,

$$c_{\text{crit}}(\alpha) \leq 2 \frac{\alpha \ln \alpha + (1 - \alpha) \ln(1 - \alpha)}{\ln(1 - \alpha^2)} \leq 2 \frac{\ln(1/\alpha) + 1}{\alpha}. \quad (3)$$

We will prove the following nearly-matching lower bound.

Theorem 1.1. *For any constant $x > 4/e$, for sufficiently small α*

$$c_{\text{crit}}(\alpha) \geq 2 \frac{\ln(1/\alpha) + 1}{\alpha} - \frac{x}{\sqrt{\alpha}}. \quad (4)$$

Coja-Oghlan and Efthymiou [5] bounded c_{crit} within a slightly larger factor $O(\sqrt{\ln(1/\alpha)/\alpha})$.

Inverting (3) and Theorem 1.1 gives the following bounds on $\alpha_{\text{crit}}(c)$. The lower bound is a significant improvement over previous results:

Corollary 1.2. *For $z > 0$, let $W(z)$ denote the unique positive root x of the equation $xe^x = z$. Then for any constant $y > 4\sqrt{2}/e$,*

$$\frac{2}{c} W\left(\frac{ec}{2}\right) - y \frac{\sqrt{\ln c}}{c^{3/2}} \leq \alpha_{\text{crit}} \leq \frac{2}{c} W\left(\frac{ec}{2}\right),$$

where the lower bound holds for sufficiently large c .

If we like we can expand $W(ec/2)$ asymptotically in c ,

$$\begin{aligned} W\left(\frac{ec}{2}\right) &= \ln c - \ln \ln c + 1 - \ln 2 + \frac{\ln \ln c}{\ln c} - \frac{1 - \ln 2}{\ln c} \\ &\quad + \frac{1}{2} \frac{(\ln \ln c)^2}{(\ln c)^2} - (2 - \ln 2) \frac{\ln \ln c}{(\ln c)^2} + \frac{3 + (\ln 2)^2 - 4 \ln 2}{2(\ln c)^2} + O\left(\frac{(\ln \ln c)^3}{(\ln c)^3}\right). \end{aligned}$$

The first few of these terms correspond to the bound in [6], and we can extract as many additional terms as we wish.

2 The weighted second moment method

Our proof uses the second moment method. For any nonnegative random variable X , the Cauchy-Schwarz inequality implies that

$$\Pr[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}. \quad (5)$$

If X counts the number of objects of a certain kind, (5) shows that at least one such object exists as long as the expected number of objects is large and the variance is not too large.

Unfortunately, applying this directly to the number X of independent sets fails utterly. The problem is that for most pairs of sets of size αn , the events that they are independent are highly correlated, unlike the case where the average degree grows sufficiently quickly with n [4, 6]. As a result, $\mathbb{E}[X^2]$ is exponentially larger than $\mathbb{E}[X]^2$, and the second moment method yields an exponentially small lower bound on $\Pr[X > 0]$.

One way to deal with these correlations, used by Frieze in [6], is to partition the vertices into sets V_i of $\lfloor 1/\alpha \rfloor$ vertices each and focus on those independent

sets that intersect each V_i exactly once. In that case, a large-deviations inequality allows us to override the correlations.

Here we pursue a different approach, inspired by the work of Achlioptas and Peres [2] on random k -SAT. The idea is to give each independent set S a weight $w(S)$, depending exponentially on local quantities in the graph. Specifically, we define

$$w(S) = \mu^{\# \text{ of edges } (u, v) \text{ with } u, v \notin S},$$

for some $\mu < 1$. If the number of edges m is fixed, the number of edges where neither endpoint is in S is simply m minus the total degree of the vertices in S . Thus we can also write

$$w(S) = \mu^{m - \sum_{v \in S} \deg(v)}. \quad (6)$$

We will apply the second moment method to the total weight of all independent sets of size αn ,

$$X = \sum_{\substack{S \subseteq V, |S| = \alpha n \\ S \text{ independent}}} w(S).$$

If we tune μ properly, then for particular α^*, c^* we have $\mathbb{E}[X^2] = \Theta(\mathbb{E}[X]^2)$, in which case $\Pr[X > 0]$ is bounded above zero. In that case $c_{\text{crit}}(\alpha^*) \geq c^*$, or equivalently $\alpha_{\text{crit}}(c^*) \geq \alpha^*$.

Why is this the right type of weight? Intuitively, one of the main sources of correlations between independent sets is the temptation to occupy low-degree vertices. For instance, any two maximal independent sets contain all the degree-zero vertices, giving them a large overlap. If X simply counts the independent sets of size αn , the resulting correlations make X 's variance exponentially large compared to the square of its expectation, and the second moment fails.

Weighting each S as in (6) counteracts this temptation, punishing sets that occupy low-degree vertices by reducing their weight exponentially. As we will see below, when μ is tuned to a particular value, making this punishment condign, these correlations disappear in the sense that the dominant contribution to $\mathbb{E}[X^2]$ comes from pairs of sets S, T of size αn such that $|S \cap T| = \alpha^2 n + O(\sqrt{n})$, just as if S and T were chosen independently from among all sets of size αn .

This is analogous to the situation for k -SAT, where satisfying assignments are correlated because of the temptation to give each variable the truth value that agrees with the majority of its literals in the formula. By giving each satisfying assignment a weight $\eta^{\# \text{ of true literals}}$ and tuning η properly, we make the dominant contribution to $\mathbb{E}[X^2]$ come from pairs of satisfying assignments which agree on $n/2 + O(\sqrt{n})$ variables, just as if they were chosen independently [2].

Proceeding, let us compute the first and second moments of our random variable X . We extend the weight function $w(S)$ to all sets $S \subseteq V$ by setting $w(S) = 0$ if S is not independent. That is,

$$X = \sum_{\substack{S \subseteq V \\ |S| = \alpha n}} w(S)$$

where

$$w(S) = \prod_{(u,v) \in E} w_{u,v}(S)$$

and

$$w_{u,v}(S) = \begin{cases} \mu & \text{if } u, v \notin S \\ 1 & \text{if } u \in S, v \notin S \text{ or vice versa} \\ 0 & \text{if } u, v \in S. \end{cases} \quad (7)$$

We start by computing $\mathbb{E}[X]$. Fix a set S of size αn . Since the m edges are chosen independently,

$$\mathbb{E}[w(S)] = w_1(\alpha, \mu)^m \quad \text{where } w_1(\alpha, \mu) = \mathbb{E}_{u,v}[w_{u,v}(S)].$$

For each edge (u, v) in $\tilde{G}(n, m)$, u and v are chosen randomly and independently, so the probabilities of the three cases in (7) are $(1 - \alpha)^2$, $2\alpha(1 - \alpha)$, and α^2 respectively. Thus

$$w_1(\alpha, \mu) = (1 - \alpha)^2 \mu + 2\alpha(1 - \alpha).$$

By linearity of expectation,

$$\mathbb{E}[X] = \sum_{\substack{S \subseteq V \\ |S| = \alpha n}} \mathbb{E}[w(S)] = \binom{n}{\alpha n} w_1(\alpha, \mu)^m.$$

Using Stirling's approximation (1) and substituting $m = cn/2$ gives

$$\mathbb{E}[X] = \Theta\left(\frac{1}{\sqrt{n}} e^{nf_1(\alpha)}\right) \quad \text{where } f_1(\alpha) = h(\alpha) + \frac{c}{2} \ln w_1(\alpha, \mu). \quad (8)$$

As before, Θ hides constant factors that depend smoothly on α .

Next we compute the second moment. We have

$$\mathbb{E}[X^2] = \mathbb{E}\left[\sum_S w(S) \sum_T w(T)\right] = \sum_{S,T} \mathbb{E}[w(S) w(T)]$$

where S and T are subsets of V of size αn . The expectation of $w(S) w(T)$ does not depend on the specific choice of S and T , but it does depend on the size of their intersection. We say that S and T have *overlap* ζ if $|S \cap T| = \zeta n$. Again using the independence of the edges, we have

$$\mathbb{E}[w(S) w(T)] = w_2(\alpha, \zeta, \mu)^m \quad \text{where } w_2(\alpha, \zeta, \mu) = \mathbb{E}_{u,v}[w_{u,v}(S) w_{u,v}(T)].$$

For each edge (u, v) of \tilde{G} , the probability that it has no endpoints in S or T is $(1 - 2\alpha + \zeta)^2$, in which case it contributes μ^2 to $w_{u,v}(S) w_{u,v}(T)$. The probability that it has one endpoint in S and none in T or vice versa is $2(2\alpha - 2\zeta)(1 - 2\alpha + \zeta)$, in which case it contributes μ . Finally, the probability that it has one endpoint

in S and one in T is $2(\alpha - \zeta)^2 + 2\zeta(1 - 2\alpha + \zeta)$, in which case it contributes 1. With the remaining probability it has both endpoints in S or T , in causing them to be non-independent and contributing zero. Thus

$$w_2(\alpha, \zeta, \mu) = (1 - 2\alpha + \zeta)^2 \mu^2 + 4(\alpha - \zeta)(1 - 2\alpha + \zeta)\mu + 2(\alpha - \zeta)^2 + 2\zeta(1 - 2\alpha + \zeta)$$

Observe that when $\zeta = \alpha^2$, as it typically would be if S and T were chosen independently and uniformly, we have

$$w_2 = w_1^2. \quad (9)$$

The number of pairs of sets S, T of size αn and intersection of size $z = \zeta n$ is the multinomial

$$\binom{n}{\zeta n, (\alpha - \zeta)n, (\alpha - \zeta)n, (1 - 2\alpha + \zeta)n} = \binom{n}{\alpha n} \binom{\alpha n}{\zeta n} \binom{(1 - \alpha)n}{(\alpha - \zeta)n},$$

and linearity of expectation gives

$$\mathbb{E}[X^2] = \sum_{z=0}^{\alpha n} \binom{n}{z, \alpha n - z, \alpha n - z, (1 - 2\alpha)n + z} w_2(\alpha, \zeta, \mu)^m.$$

This sum is dominated by the terms where $\zeta = z/n$ is bounded inside the interval $(0, \alpha)$. Stirling's approximation then gives

$$\binom{n}{\zeta n, (\alpha - \zeta)n, (\alpha - \zeta)n, (1 - 2\alpha + \zeta)n} = \Theta\left(\frac{e^{n[h(\alpha) + \alpha h(\zeta/\alpha) + (1 - \alpha)h(\frac{\alpha - \zeta}{1 - \alpha})]}}{n^{3/2}}\right),$$

where Θ hides constants that vary slowly with α and ζ . Thus the contribution to $\mathbb{E}[X^2]$ of pairs of sets with overlap $\zeta \in (0, \alpha)$ is

$$\frac{1}{n^{3/2}} e^{nf_2(\alpha, \zeta, \mu)} \quad (10)$$

where

$$f_2(\alpha, \zeta, \mu) = h(\alpha) + \alpha h\left(\frac{\zeta}{\alpha}\right) + (1 - \alpha)h\left(\frac{\alpha - \zeta}{1 - \alpha}\right) + \frac{c}{2} \ln w_2(\alpha, \zeta, \mu).$$

Combining (10) with (8), we can write

$$\frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} = \Theta\left(\frac{1}{\sqrt{n}} \sum_{z=0}^{\alpha n} e^{n\phi(z/n)}\right), \quad (11)$$

where

$$\begin{aligned} \phi(\zeta) &= f_2(\alpha, \zeta, \mu) - 2f_1(\alpha, \mu) \\ &= \alpha h\left(\frac{\zeta}{\alpha}\right) + (1 - \alpha)h\left(\frac{\alpha - \zeta}{1 - \alpha}\right) - h(\alpha) + \frac{c}{2} \ln \frac{w_2(\alpha, \zeta, \mu)}{w_1(\alpha, \mu)^2}. \end{aligned}$$

Using (9) and the fact that the entropy terms cancel, we have

$$\phi(\alpha^2) = 0.$$

In other words, the contribution to $\mathbb{E}[X^2]$ from pairs of sets with overlap α^2 is proportional to $\mathbb{E}[X]^2$.

We can now replace the sum in (11) with an integral,

$$\frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} = \Theta\left(\frac{1}{\sqrt{n}} \sum_{z=0}^{\alpha n} e^{n\phi(z/n)}\right) = \Theta\left(\sqrt{n} \int_0^\alpha e^{n\phi(\zeta)} d\zeta\right),$$

and evaluate this integral using Laplace's method as in [1, Lemma 3]. Its asymptotic behavior depends on the maximum value of ϕ ,

$$\phi_{\max} = \max_{\zeta \in [0, \alpha]} \phi(\zeta).$$

If $\phi'' < 0$ at the corresponding ζ_{\max} , then it is dominated by an interval of width $\Theta(1/\sqrt{n})$ around ζ_{\max} and

$$\frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} = \Theta(e^{n\phi_{\max}}).$$

If $\phi_{\max} = \phi(\alpha^2) = 0$, then $\mathbb{E}[X^2] = \Theta(\mathbb{E}[X]^2)$ and the second moment method succeeds. Thus our goal is to show that ϕ is maximized at α^2 .

For this to happen, we at least need $\zeta = \alpha^2$ to be a *local* maximum of ϕ . In particular, we need

$$\phi'(\alpha^2) = 0. \tag{12}$$

Differentiating, we find that (12) holds if

$$\mu = \frac{1-2\alpha}{1-\alpha}.$$

Henceforth, we will fix μ to this value. In that case we have

$$w_1 = 1 - \alpha \quad \text{and} \quad w_2 = (1 - \alpha)^2 + \frac{(\zeta - \alpha^2)^2}{(1 - \alpha)^2},$$

so

$$\phi(\zeta) = \alpha h\left(\frac{\zeta}{\alpha}\right) + (1 - \alpha) h\left(\frac{\alpha - \zeta}{1 - \alpha}\right) - h(\alpha) + \frac{c}{2} \ln\left(1 + \frac{(\zeta - \alpha^2)^2}{(1 - \alpha)^4}\right)$$

The remainder of this paper is dedicated to showing that for sufficiently small α as a function of c or vice versa, ϕ is indeed maximized at α^2 .

3 Finding and bounding the maxima

Using $\ln(1+x) \leq x$, we write $\phi(\zeta) \leq \psi(\zeta)$ where

$$\psi(\zeta) = \alpha h\left(\frac{\zeta}{\alpha}\right) + (1-\alpha) h\left(\frac{\alpha-\zeta}{1-\alpha}\right) - h(\alpha) + \frac{c}{2} \frac{(\zeta-\alpha^2)^2}{(1-\alpha)^4}. \quad (13)$$

Note that

$$\psi(\alpha^2) = \phi(\alpha^2) = 0.$$

Our goal is to show for an appropriate c that $\zeta = \alpha^2$ is in fact the global maximum of ψ , and therefore of ϕ . In what follows, asymptotic symbols such as O and o refer to the limit $\alpha \rightarrow 0$, or equivalently the limit $c \rightarrow \infty$. Error terms may be positive or negative unless otherwise stated.

The first two derivatives of $\psi(\zeta)$ are

$$\psi'(\zeta) = \frac{c(\zeta-\alpha^2)}{(1-\alpha)^4} + 2\ln(\alpha-\zeta) - \ln\zeta - \ln(1-2\alpha+\zeta) \quad (14)$$

$$\psi''(\zeta) = \frac{c}{(1-\alpha)^4} - \frac{2}{\alpha-\zeta} - \frac{1}{\zeta} - \frac{1}{1-2\alpha+\zeta} \quad (15)$$

The second derivative $\psi''(\zeta)$ tends to $-\infty$ at $\zeta = 0$ and $\zeta = \alpha$. Setting $\psi''(\zeta) = 0$ yields a cubic equation in ζ which has one negative root and, for sufficiently small α , two positive roots in the interval $[0, \alpha]$. Thus for each α and sufficiently small α , there are $0 < \zeta_1 < \zeta_2 < \alpha$ where

$$\psi''(\zeta) \begin{cases} < 0 & 0 < \zeta < \zeta_1 \\ > 0 & \zeta_1 < \zeta < \zeta_2 \\ < 0 & \zeta_2 < \zeta < \alpha. \end{cases}$$

It follows that ψ can have at most two local maxima. One is in the interval $[0, \zeta_1]$, and the following lemma shows that for the relevant α and c this is α^2 :

Lemma 3.1. *If $c = o(1/\alpha^2)$ then for sufficiently small α , $\zeta_1 > \alpha^2$ and $\psi(\alpha^2)$ is a local maximum.*

The other local maximum is in the interval $[\zeta_2, \alpha]$, and we denote it ζ_3 . To locate it, first we bound ζ_2 :

Lemma 3.2. *If*

$$c = (2 + o(1)) \frac{\ln(1/\alpha)}{\alpha},$$

then

$$\frac{\zeta_2}{\alpha} = 1 - \delta_2 \quad \text{where} \quad \delta_2 = \frac{1 + o(1)}{\ln(1/\alpha)}.$$

Thus ζ_2/α , and therefore ζ_3/α , tends toward 1 as $\alpha \rightarrow 0$.

We can now locate ζ_3 when α is close to its critical value.

Lemma 3.3. *If*

$$c = \frac{1}{\alpha} (2 \ln(1/\alpha) + 2 - o(1)),$$

then

$$\frac{\zeta_3}{\alpha} = 1 - \delta_3 \quad \text{where} \quad \delta_3 = \frac{1 + o(1)}{e} \sqrt{\alpha}.$$

Lemma 3.4. *For any constant $x > 4/e$, if*

$$c = \frac{2 \ln(1/\alpha) + 2 - x\sqrt{\alpha}}{\alpha},$$

then $\psi(\zeta_3) < 0$ for sufficiently small α .

4 Proofs

Proof of Lemma 3.1. Setting $\zeta = \alpha^2$ in (15) gives

$$\psi''(\alpha^2) < \frac{c}{(1-\alpha)^4} - \frac{1}{\alpha^2}.$$

If $c = o(1/\alpha^2)$ this is negative for sufficiently small α , in which case $\zeta_1 > \alpha^2$ and $\psi(\alpha^2)$ is a local maximum. \square

Proof of Lemma 3.2. For any constant b , if

$$\frac{\zeta}{\alpha} = 1 - \delta \quad \text{where} \quad \delta = \frac{b}{\ln(1/\alpha)} \tag{16}$$

then (15) gives

$$\psi''(\zeta) = \left(2 - \frac{2}{b} + o(1)\right) \frac{\ln(1/\alpha)}{\alpha} - O(1/\alpha).$$

If $b \neq 1$, for sufficiently small α this is negative if $b < 1$ and positive if $b > 1$. Therefore $\zeta_2/\alpha = 1 - \delta_3$ where $\delta_3 = (1 + o(1))/\ln(1/\alpha)$. \square

Proof of Lemma 3.3. Lemma 3.2 tells us that $\zeta_3 = \alpha(1 - \delta)$ for some

$$\delta < \frac{1 + o(1)}{\ln(1/\alpha)}.$$

Setting $\zeta = \alpha(1 - \delta)$ in (14) and using

$$\frac{1}{(1-\alpha)^4} = 1 + O(\alpha) \quad \text{and} \quad -\ln(1-x) = O(x)$$

gives, after some algebra,

$$\psi'(\zeta) = \alpha c + \ln \alpha + 2 \ln \delta + O(\alpha \delta c) + O(\alpha^2 c).$$

For any constant b , setting

$$\delta = \frac{b\sqrt{\alpha}}{e}$$

gives

$$\psi'(\zeta) = \alpha c + 2 \ln \alpha + 2 \ln b - 2 + O(\alpha^{3/2} c),$$

and setting

$$c = \frac{2 \ln(1/\alpha) + 2 - \varepsilon}{\alpha}$$

then gives

$$\psi'(\zeta) = 2 \ln b - \varepsilon + o(1).$$

If $\varepsilon = o(1)$ and $b \neq 1$, for sufficiently small α this is negative if $b < 1$ and positive if $b > 1$. Therefore $\zeta_3/\alpha = 1 - \delta_3$ where $\delta_3 = (1 + o(1))\sqrt{a}/e$. \square

Proof of Lemma 3.4. Setting $\zeta = \alpha(1 - \delta)$ where $\delta = b\sqrt{a}/e$ in (13) and using the Taylor series

$$\frac{1}{(1 - \alpha)^4} = 1 + 4\alpha + O(\alpha^2) \quad \text{and} \quad -\ln(1 - x) = x + x^2/2 + O(x^3)$$

gives, after a fair amount of algebra,

$$\begin{aligned} \psi(\zeta) &= \alpha(\ln \alpha - 1) - \left(\frac{2b \ln \alpha - 4b + 2b \ln b}{e} \right) \alpha^{3/2} + \left(\frac{c+1}{2} - \frac{b^2}{2e^2} \right) \alpha^2 \\ &\quad - \frac{b}{e} \alpha^{5/2} c + \left(\frac{b^2}{2e^2} + 1 \right) \alpha^3 c + O(\alpha^{7/2} c) + O(\alpha^{5/2}). \end{aligned}$$

Setting

$$c = \frac{2 \ln(1/\alpha) + 2 - x\sqrt{\alpha}}{\alpha}$$

for constant x causes the terms proportional to $\alpha \ln \alpha$, α , and $\alpha^{3/2} \ln \alpha$ to cancel, leaving

$$\psi(\zeta) = \left(\frac{2b(1 - \ln b)}{e} - \frac{x}{2} \right) \alpha^{3/2} + O(\alpha^2).$$

The coefficient of $\alpha^{3/2}$ is maximized when $b = 1$, and is negative whenever $x > 4/e$. In that case, $\psi(\zeta_3) < 0$ for sufficiently small α , completing the proof. \square

Proof of Corollary 1.2. First note that

$$\alpha_0 = \frac{2}{c} W\left(\frac{ec}{2}\right) \tag{17}$$

is the root of the equation

$$c = 2 \frac{\ln(1/\alpha_0) + 1}{\alpha_0},$$

since we can also write it as

$$e^{c\alpha/2} = \frac{e}{\alpha_0},$$

and multiplying both sides by $c\alpha_0/2$ gives

$$\frac{c\alpha_0}{2} e^{c\alpha_0/2} = \frac{ec}{2},$$

in which case (17) follows from the definition of W .

The root α of

$$c = 2 \frac{\ln(1/\alpha) + 1}{\alpha} - \frac{x}{\sqrt{\alpha}}$$

is then at least

$$\frac{2}{c} W\left(\frac{ec}{2}\right) + (x + o(1)) \frac{\partial \alpha_0}{\partial c} \sqrt{\frac{c}{2 \ln c}}$$

since $\alpha = (1 + o(1))2 \ln c/c$ and $\partial^2 \alpha_0 / \partial^2 c \geq 0$. Since

$$\frac{\partial \alpha_0}{\partial c} = -(1 + o(1)) \frac{2 \ln c}{c^2},$$

the statement follows from Theorem 1.1. \square

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